

A complete solution of Samuel's problem

Marcos Dajczer and Ruy Tojeiro

Abstract

We give a complete solution of a problem in submanifold theory posed and partially solved by the eminent algebraic geometer Pierre Samuel in 1947. Namely, to determine all pairs of immersions $f, g: M^n \rightarrow \mathbb{R}^N$ into Euclidean space that have the same Gauss map and induce conformal metrics on the manifold M^n . The case of isometric induced metrics was solved in 1985 by the first author and D. Gromoll.

1 Introduction

To what extent is a surface $f: M^2 \rightarrow \mathbb{R}^3$ determined by its conformal structure and its Gauss map? This problem was studied back in 1867 by Christoffel [1], who found all local exceptions. Besides minimal surfaces, the only remaining surfaces admitting nontrivial conformal deformations preserving the Gauss map are isothermic surfaces, which are characterized by carrying local conformal parameterizations by curvature lines on the open subset of nonumbilic points.

For Euclidean surfaces of arbitrary codimension, the problem has been studied by several geometers [8], [11], [14], [15] and [18]. The article [15] goes back to 1947 and was the first publication by the eminent algebraic geometer Pierre Samuel. He showed that exceptions are again minimal surfaces and a natural generalization of isothermic surfaces, according as the deformation preserves or reverses orientation, respectively. His result was totally or partially rediscovered in the other papers much later.

By the above, a surface in \mathbb{R}^N with nonvanishing mean curvature vector admits no nontrivial orientation-preserving conformal deformation preserving the Gauss map. In fact, in [11] a representation theorem is given for any locally conformal map of a Riemann surface $f: M^2 \rightarrow \mathbb{R}^N$ with nonvanishing mean curvature vector in terms of its Gauss map with values in the quadric $\mathbb{Q}^{N-2} \subset \mathbb{CP}^{N-1}$. The case of minimal surfaces is quite different since the Gauss map is only part of the data in the generalized Weierstrass parametrization given in [10].

The general problem of looking for all pairs of immersions $f, g: M^n \rightarrow \mathbb{R}^N$ into Euclidean space that have the same Gauss map into the Grassmannian $G_{N,n}$ and induce conformal metrics on M^n was also considered by Samuel [15]. He divided his study in

two cases, called holonomic and nonholonomic according as some natural distributions that arise are integrable or not. Samuel gave a complete solution of the problem in the holonomic case for analytic immersions. However, he was not able to obtain a full classification in the nonholonomic case, probably because several of the necessary tools in submanifold theory were not fully developed at that time. On the other hand, his idea of working with the complexified tensors related to the problem turns out to be very efficient and is also the starting point of our approach in this paper.

The isometric version of the problem was solved by the first author and Gromoll [3] (see also [12]). Namely, what are all pairs of immersions $f, g: M^n \rightarrow \mathbb{R}^N$ that induce the same metric on M^n and have the same Gauss map? Locally, solutions are (products of) real minimal Kaehler submanifolds, which admit associated families as minimal surfaces. Globally, the family of noncongruent isometric immersions $g: M^n \rightarrow \mathbb{R}^N$ with the same Gauss map as a given isometric immersion $f: M^n \rightarrow \mathbb{R}^N$ is parametrized by a compact abelian group whose structure was determined.

For hypersurfaces of dimension $n \geq 3$, the problem was considered by the first author and Vergasta [6]. In this case, the only exceptions (hypersurfaces admitting conformal non-isometric and not conformally congruent deformations preserving the Gauss map) are rotation hypersurfaces over plane curves and minimal surfaces in \mathbb{R}^3 . For the proof, the authors made strong use of Cartan's criterion for conformal rigidity of hypersurfaces, namely, an Euclidean hypersurface must have a principal curvature of multiplicity at least $n - 2$ in order to admit nontrivial conformal deformations. Therefore, most of the arguments in [6] can not be extended for submanifolds of higher codimension.

Recently, a special case of the problem was studied in [16] as one of the approaches to look for higher dimensional analogues of isothermic surfaces. However, that case is comprised in the holonomic case of the problem solved by Samuel (although stated in a rather different way) of whose work the second author was unaware at that time.

In this paper we provide a complete solution of Samuel's problem. Surprisingly enough, there are few examples of submanifolds that admit conformal non-isometric deformations preserving the Gauss map. First, one can take a cone over a spherical submanifold and consider its image under an inversion with respect to the center of the sphere. Since the Gauss map is constant along the rulings and these are preserved by the inversion, the deformation is conformal and preserves the Gauss map. Start now with a minimal real Kaehler cone and perform the preceding deformation after isometrically deforming it with preservation of the Gauss map. Then, one obtains a conformal non-isometric deformation that preserves the Gauss map but does not leave the submanifold invariant. We point out that any minimal real Kaehler cone is the real part of a holomorphic isometric immersion in \mathbb{C}^N obtained as the lifting of a holomorphic isometric immersion into \mathbb{CP}^{N-1} .

Apart from the above examples with somewhat trivial deformations in the conformal realm, all remaining ones of dimension $n \geq 3$ are built up from either curves or minimal surfaces by making warped products of them (in the sense of [13]; see Sec-

tion 5.1 for details) with spherical submanifolds. These include cones as well as the rotational hypersurfaces described in [6] as particular cases. However, there appears an interesting example that can not occur as a hypersurface. Namely, a triply warped product submanifold having as profile a degenerate minimal surface in the sense of [10] (see Proposition 20 and Remark 21 below).

The paper is organized as follows. In Section 2 we derive some basic properties of pairs of immersions $f, g: M^n \rightarrow \mathbb{R}^N$ that have the same Gauss map into the Grassmannian manifold of nonoriented n -planes in \mathbb{R}^N . These properties are combined in Section 3 with the relation between the Levi-Civita connections of conformal metrics to give a proof of a basic lemma due to Vergasta [18]. It states that conformal Gauss-map-preserving deformations of a submanifold $f: M^n \rightarrow \mathbb{R}^N$ are determined by pairs (T, φ) satisfying a certain differential equation, where T is an orthogonal tensor and φ is a smooth function on M^n . The complexified version of this equation is the basic tool in our solution of the problem.

Section 4 is devoted to the surface case, which plays a key role in the solution of the general case. In Section 5, we present the nontrivial examples of pairs of conformal immersions $f, g: M^n \rightarrow \mathbb{R}^N$, $n \geq 3$, with the same Gauss map. In the following section we introduce some further tools and derive basic lemmata that are used in the last section in order to show that such examples comprise all possible ones. This is done by a case-by-case study of the various possibilities for the splitting of the complexified tangent bundle of the manifold into eigenbundles of the corresponding orthogonal tensor T .

2 Immersions with the same Gauss map

In this section, we discuss basic facts about pairs of immersions having the same Gauss map, but make no assumptions whatsoever on their induced metrics.

The *Gauss map* into the Grassmann manifold $G_{N,n}$ of unoriented n -planes in \mathbb{R}^N of a given immersion $f: M^n \rightarrow \mathbb{R}^N$ assigns to each $p \in M^n$ the tangent space f_*T_pM . That another immersion $g: M^n \rightarrow \mathbb{R}^N$ has the same Gauss map as f is equivalent to the existence of a tensor $\Phi \in C^\infty(T^*M \otimes TM)$ such that

$$g_* = f_* \circ \Phi.$$

It was observed in [5] that Φ has the following properties.

Proposition 1. *The following holds:*

(i) Φ is a Codazzi tensor, i.e.,

$$(\nabla_X \Phi)Y = (\nabla_Y \Phi)X \quad \text{for all } X, Y \in TM.$$

(ii) The second fundamental form α_f of f commutes with Φ , i.e.,

$$\alpha_f(X, \Phi Y) = \alpha_f(\Phi X, Y) \quad \text{for all } X, Y \in TM.$$

Conversely, if $\Phi \in C^\infty(T^*M \otimes TM)$ satisfies (i) and (ii) and M^n is simply connected, then there exists an immersion $g: M^n \rightarrow \mathbb{R}^N$ such that $g_* = f_* \circ \Phi$.

Proof: Regard $\omega = f_* \circ \Phi$ as a one-form on M^n with values in \mathbb{R}^N . Then,

$$d\omega(X, Y) = f_*(\nabla_X \Phi Y - \nabla_Y \Phi X - \Phi[X, Y]) + \alpha_f(X, \Phi Y) - \alpha_f(\Phi X, Y). \quad \blacksquare$$

Moreover, we have the following relations.

Proposition 2. *The Levi-Civita connections of the induced metrics and the second fundamental forms of f and g are related by*

$$\Phi \tilde{\nabla}_X Y = \nabla_X \Phi Y \tag{1}$$

and

$$\alpha_g(X, Y) = \alpha_f(\Phi X, Y) \quad \text{for all } X, Y \in TM.$$

Proof: Both assertions follow from

$$f_* \nabla_X \Phi Y + \alpha_f(\Phi X, Y) = \bar{\nabla}_X f_* \Phi Y = \bar{\nabla}_X g_* Y = f_* \Phi \tilde{\nabla}_X Y + \alpha_g(X, Y),$$

where $\bar{\nabla}$ stands for the derivative in \mathbb{R}^N . \blacksquare

3 Vergasta's basic lemma

Next, we give a proof of a basic fact due to Vergasta [18] and discuss its complexified version.

In addition to $f, g: M^n \rightarrow \mathbb{R}^{n+p}$ having the same Gauss map, we assume that they are conformal, i.e., there exists $\varphi \in C^\infty(M)$ so that the induced metrics are related by

$$\langle \cdot, \cdot \rangle_g = e^{2\varphi} \langle \cdot, \cdot \rangle_f,$$

where e^φ is called the *conformal factor* of $\langle \cdot, \cdot \rangle_g$ with respect to $\langle \cdot, \cdot \rangle_f$. In this case,

$$T = e^{-\varphi} \Phi$$

is an orthogonal tensor with respect to $\langle \cdot, \cdot \rangle_f$.

The following lemma due to Vergasta [18] is the starting point of our solution of Samuel's problem.

Lemma 3. *The pair (T, φ) satisfies the differential equation*

$$(\nabla_X T)Y = \langle Y, \nabla \varphi \rangle TX - \langle X, Y \rangle T \nabla \varphi \quad \text{for all } X, Y \in TM. \quad (2)$$

Conversely, for a given isometric immersion $f: M^n \rightarrow \mathbb{R}^{n+p}$ of a simply connected Riemannian manifold, any pair (T, φ) satisfying (2) and

$$\alpha_f(TX, Y) = \alpha_f(X, TY) \quad \text{for all } X, Y \in TM,$$

gives rise to a conformal immersion $g: M^n \rightarrow \mathbb{R}^{n+p}$ with the same Gauss map.

Proof: The Levi-Civita connections for the conformal induced metrics relate as

$$\tilde{\nabla}_X Y = \nabla_X Y + X(\varphi)Y + Y(\varphi)X - \langle X, Y \rangle \nabla \varphi. \quad (3)$$

On the other hand, we obtain using (1) that

$$T\tilde{\nabla}_X Y = e^{-\varphi} \Phi \tilde{\nabla}_X Y = e^{-\varphi} \nabla_X \Phi Y = e^{-\varphi} \nabla_X e^\varphi TY = X(\varphi)TY + \nabla_X TY, \quad (4)$$

and the claim follows by comparing (3) and (4).

The converse follows from the converse statement of Proposition 1, after checking that $\Phi = e^\varphi T$ is a Codazzi tensor if (T, φ) satisfies (2). ■

3.1 The complexified orthogonal tensor

Given an immersion $f: M^n \rightarrow \mathbb{R}^N$, we may extend the induced metric and the second fundamental form to complex bilinear forms

$$\langle \cdot, \cdot \rangle : (TM \otimes \mathbb{C}) \times (TM \otimes \mathbb{C}) \rightarrow \mathbb{C} \quad \text{and} \quad \alpha_f : (TM \otimes \mathbb{C}) \times (TM \otimes \mathbb{C}) \rightarrow (T^\perp M \otimes \mathbb{C}).$$

Let $g: M^n \rightarrow \mathbb{R}^N$ be another immersion with $g_* = f_* \circ e^\varphi T$, where $\varphi \in C^\infty(M)$ and T is an orthogonal tensor on M^n . Then, all eigenvalues of the complex linear extension of T have length one and we can pointwise decompose $TM \otimes \mathbb{C}$ as

$$TM \otimes \mathbb{C} = L_+ \oplus L_- \oplus L_c$$

with $L_\pm = E_{\pm 1} = \ker(T \mp I)$ and $L_c = \sum_{i=1}^k (E_{\lambda_i} \oplus E_{\bar{\lambda}_i})$, where $(\lambda_i, \bar{\lambda}_i)$ are the distinct pairs of complex-conjugate eigenvalues, $E_{\lambda_i} = \ker(T - \lambda_i I)$ and $E_{\bar{\lambda}_i} = \ker(T - \bar{\lambda}_i I)$.

Lemma 4. *The eigenspaces of T satisfy*

$$(i) \quad \langle E_\lambda, E_\mu \rangle = 0 \quad \text{unless } \mu = \bar{\lambda} = 1/\lambda,$$

$$(ii) \quad \alpha_f(E_\lambda, E_\mu) = 0 \quad \text{unless } \mu = \lambda.$$

Proof: For any $U \in E_\lambda$ and $V \in E_\mu$, we have $\langle U, V \rangle = \langle TU, TV \rangle = \lambda\mu\langle U, V \rangle$ and $\lambda\alpha_f(U, V) = \alpha_f(TU, V) = \alpha_f(U, TV) = \mu\alpha_f(U, V)$, and the result follows. ■

In this paper we mostly work with the complexified version of (2), that is,

$$(\nabla_U T)V = \langle V, \nabla\varphi \rangle TU - \langle U, V \rangle T\nabla\varphi \quad \text{for all } U, V \in TM \otimes \mathbb{C}, \quad (5)$$

where ∇ is also extended complex bilinearly. For convenience, we give next how the equation reads when applied to particular pairs of eigenvectors of T .

Lemma 5. *The following equations hold:*

$$(T - \mu I)\nabla_Z W = Z(\mu)W - \lambda W(\varphi)Z + \langle Z, W \rangle T\nabla\varphi \quad \text{for all } Z \in E_\lambda, W \in E_\mu, \quad (6)$$

$$(T - \lambda I)\nabla_X Z = X(\lambda)Z - Z(\varphi)X \quad \text{for all } X \in E_+, Z \in E_\lambda. \quad (7)$$

The reader should keep in mind the following simple but useful fact.

Fact 6. A pair (T, φ) satisfies (2) or (5) if and only if the same holds for the pair $(-T, \varphi)$. Since the eigenbundles L_+ and L_- are interchanged for T and $-T$, any assertion on L_+ is also valid for L_- just by applying it to $-T$.

4 The surface case

This section is devoted to review the results in the case of surfaces with arbitrary codimension [11], [14], [15] which will play an important role in the study of the general case.

There are three possibilities for the tensor T :

Case 1. $TM \otimes \mathbb{C} = L_+$. Equation (5) yields $\nabla\varphi = 0$, and we conclude that g is the composition of f with a homothety and a translation.

Case 2. $TM \otimes \mathbb{C} = L_+ \oplus L_-$. Let X and Y be unit vector fields spanning L_+ and L_- , respectively. Set

$$\eta_+ = \nabla_X X \quad \text{and} \quad \eta_- = \nabla_Y Y.$$

Then (5) reduces to the system of equations

$$X(\varphi) = 2\langle \eta_-, X \rangle \quad \text{and} \quad Y(\varphi) = 2\langle \eta_+, Y \rangle \quad (8)$$

whose integrability condition is $\langle \nabla_X \eta_+, Y \rangle = \langle \nabla_Y \eta_-, X \rangle$. But this is precisely the condition for the existence of local isothermal coordinates whose coordinate curves are tangent to X and Y (see [7]-III, (36) in p. 154, or Theorem 4.3 in [17]). Since $\alpha_f(X, Y) = 0$ by Lemma 4-(ii), then X and Y are principal directions and thus the surface has flat normal bundle. Hence, it is an isothermic surface. Conversely, any simply connected

isothermic surface has exactly one conformal deformation with the same Gauss map, called its *dual isothermic surface*.

Case 3. $TM \otimes \mathbb{C} = L_c = E_\lambda \oplus E_{\bar{\lambda}}$ with $\lambda = e^{i\theta}$. Equation (5) reduces to

$$\bar{\lambda}Z(\lambda) = Z(\varphi), \text{ for all } Z \in TM \otimes \mathbb{C}. \quad (9)$$

Choose local isothermal coordinates (u, v) with coordinates vector fields $\{\partial u, \partial v\}$ and set $Z = \partial/\partial z = (1/2)(\partial u - i\partial v)$. Then (9) is equivalent to the functions φ and θ being harmonic conjugate. Moreover, $\alpha(Z, \bar{Z}) = 0$ says that f is a minimal surface. Let M^2 be simply connected with global isothermal coordinates (u, v) . Then, the family of its conformal deformations with the same the Gauss map is in correspondence with the set of holomorphic functions $\psi = \varphi + i\theta$. The element of the family corresponding to ψ is the minimal surface

$$g = \int e^\psi f_z dz. \quad (10)$$

Remark 7. If $\nabla\varphi = 0$ in Case 1, then (8) implies that the integral curves of X and Y are geodesics. Since $\alpha_f(X, Y) = 0$, it follows that $f = \alpha \times \beta$ is a product of curves whereas $g = \alpha \times (-\beta)$, up to homothety and translation. In Case 3, that $\nabla\varphi = 0$ forces λ to be constant, and thus f and g are members of an associated family of minimal surfaces, up to homothety.

4.1 Deformations preserving the hyperbolic metric

For later use, we study the Gauss map preserving deformations $g = (\alpha^1, \dots, \alpha^{m-1}, \alpha)$ of a minimal surface $f = (a^1, \dots, a^{m-1}, a)$ with $a > 0$, i.e., contained in the upper half-space \mathbb{R}_+^m , that preserve the metric induced from the hyperbolic metric on \mathbb{R}_+^m .

That f and g induce the same metric from the hyperbolic metric on \mathbb{R}_+^m means that they induce conformal metrics from the Euclidean metric on \mathbb{R}_+^m with conformal factor e^φ satisfying $e^\varphi a = \alpha$. Differentiating this equation and using that $\psi = \varphi + i\theta$ is holomorphic gives

$$\alpha_z = e^\varphi(a_z + a\varphi_z) = e^\varphi(a_z + ia\theta_z).$$

From (10) we have $\alpha_z = e^\psi a_z$. We obtain that

$$\theta_z = (1 - e^{i\theta}) \frac{ia_z}{a}.$$

The latter can be written as $((e^{i\theta} - 1)/a)_{\bar{z}} = 0$. Thus, it is equivalent to $(e^{i\theta} - 1)/a$ being a holomorphic function, say $k = u + iv$.

From $e^{i\theta} = 1 + au + iav$, we obtain $(1 + au)^2 + (av)^2 = 1$. Thus $a = -2u/(u^2 + v^2)$. Hence, a is the real part of the holomorphic function $-2/k$, and therefore $k = -2/A$ where $A = a + i\bar{a}$ is holomorphic. It follows that

$$e^{i\theta} = ak + 1 = -\bar{A}^2/|A|^2.$$

Therefore, the holomorphic functions e^ψ and $-1/A^2$ coincide, since they have the same argument. Moreover, if $\mathcal{A} = \alpha + i\bar{\alpha}$ is holomorphic, then this and $\alpha_z = e^\psi a_z$ yield $\mathcal{A}_z = e^\psi A_z = -A_z/A^2 = (1/A)_z$, hence α is the real part of $1/A$, up to a constant.

We summarize the preceding discussion in the following statement.

Proposition 8. *Two minimal surfaces $f, g: L^2 \rightarrow \mathbb{R}_+^m$ have the same (oriented) Gauss map and are isometric with respect to the hyperbolic metric on \mathbb{R}_+^m and if and only if they relate as follows: if f is parametrized in isothermal coordinates by $f = (a^1, \dots, a^{m-1}, a)$ and $A = a + i\bar{a}$ is holomorphic, then*

$$g = - \int \frac{1}{A^2} f_z dz. \quad (11)$$

Moreover, the last coordinate function of g is the real part of $1/A$.

Remark 9. If $f: L^2 \rightarrow \mathbb{R}_+^m$ is totally geodesic, endowing L^2 with the metric induced by the hyperbolic metric on \mathbb{R}_+^m we have that L^2 is either an open subset of the Euclidean plane \mathbb{R}^2 or of the hyperbolic plane $\mathbb{H}_c^2, c \in [-1, 0)$, according as $f_0(L^2)$ is parallel to the boundary of \mathbb{R}_+^m or not. Regard f as the restriction to L^2 of an isometric immersion of either \mathbb{R}^2 or \mathbb{H}_c^2 into \mathbb{H}^m , respectively. Then g is given, up to a translation, by $g = f \circ h$, where h is the restriction to L^2 of an isometry of \mathbb{R}^2 or \mathbb{H}_c^2 , respectively.

5 The general case

To start the study of the general problem considered by Samuel we present in this section several families of examples.

5.1 A trivial example

Example 10. Let $f: U \subset \mathbb{R}^m \rightarrow \mathbb{R}^N$ be totally geodesic and let $\phi: U \rightarrow U$ be a conformal diffeomorphism. Then f and $g = f \circ \phi$ are conformal immersions with the same Gauss map.

For an f as above we have the following fact.

Proposition 11. *If U is simply connected, then any immersion $g: U \rightarrow \mathbb{R}^N$ that is conformal to f and has the same Gauss map is given in this way.*

Proof: Write $g_* = f_* \circ \Phi$ for $\Phi \in C^\infty(T^*U \otimes TU)$. We regard Φ as a one-form in U with values in \mathbb{R}^n . Then, being a Codazzi tensor is equivalent to being closed, hence exact. ■

5.2 Minimal real Kaehler cones

By a *real Kaehler* submanifold we mean an isometric immersion $f: M^n \rightarrow \mathbb{R}^N$ of a Kaehler manifold (M^n, J) . Here n stands for the real dimension. It was shown in [4] (see also [12]) that any such minimal f is *pseudo-holomorphic*. This means that its second fundamental form commutes with the complex structure, i.e.,

$$\alpha_f(X, JY) = \alpha_f(JX, Y) \quad \text{for all } X, Y \in TM.$$

Clearly, for each $\theta \in [0, 2\pi)$ the tensor $J_\theta = \cos \theta I + \sin \theta J$ is parallel. If M^n is simply connected, by Proposition 1, there exists an isometric immersion $f_\theta: M^n \rightarrow \mathbb{R}^N$ such that $f_{\theta*} = f_* \circ J_\theta$. Moreover, by Proposition 2, the second fundamental form α_θ of f_θ is given by

$$\alpha_\theta(X, Y) = \alpha_f(J_\theta X, Y) \quad \text{for all } X, Y \in TM. \quad (12)$$

Thus f_θ is also pseudo-holomorphic, hence minimal. Therefore, any simply connected minimal real Kaehler submanifold $f: M^n \rightarrow \mathbb{R}^N$ comes (like minimal surfaces) with its *associated family* of minimal isometric immersions f_θ , all having the same Gauss map. Moreover, the family is trivial (all f_θ are congruent to f) if and only if (N is even and) f is holomorphic, that is, $f_* \circ J = \tilde{J} \circ f_*$, where \tilde{J} is a complex structure of \mathbb{R}^N . In particular, any minimal isometric immersion $f: M^n \rightarrow \mathbb{R}^N$ of a simply connected Kaehler manifold is the real part of a holomorphic isometric immersion $F: M^n \rightarrow \mathbb{C}^N$. In fact, the map

$$F = (1/\sqrt{2})(f + if_{\pi/2}): M^n \rightarrow \mathbb{C}^N = \mathbb{R}^N \oplus i\mathbb{R}^N \quad (13)$$

is isometric and holomorphic (see [3]).

Recall that the *relative nullity* distribution of an isometric immersion $f: M^n \rightarrow \mathbb{R}^N$ assigns to each point of M^n the tangent subspace

$$\Delta_f = \{X \in TM : \alpha_f(X, Y) = 0 \text{ for all } Y \in TM\}.$$

If f is pseudo-holomorphic, then Δ_f is J -invariant. In particular, $\Delta_{f_\theta} = \Delta_f$ from (12).

In this paper, we focus in the case in which f is also a cone, that is, admits a foliation by straight lines through a common point of \mathbb{R}^N . The next result shows how any such example arises.

Proposition 12. *Let $f: M^n \rightarrow \mathbb{R}^N$, $n \geq 4$, be a minimal isometric immersion of a simply connected Kaehler manifold. Then f is a cone if and only if f is the real part of a holomorphic isometric immersion $F: M^n \rightarrow \mathbb{C}^N$ obtained as the lifting of a holomorphic immersion $\bar{f}: M^{n-2} \rightarrow \mathbb{CP}^{N-1}$ by the projection $\pi: \mathbb{C}^N \rightarrow \mathbb{CP}^{N-1}$.*

Proof: We prove the direct statement, since the converse is clear. We already know that F in (13) is isometric and holomorphic, where $g := f_{\pi/2}$ is the conjugate immersion to

f , i.e., $g_* = f_* \circ J$. Since f is a cone, there exists a unit vector field R and a smooth function γ on M^n such that the map $h = f + \gamma^{-1}f_*R$ is constant. It suffices to show that the map

$$\ell = g + \gamma^{-1}g_*R$$

is also constant and that $L = \text{span}\{R, JR\}$ is an integrable distribution whose leaves are mapped by f and g into affine planes of \mathbb{R}^N . Then, the images by F of the leaves of L give rise to a foliation of $F(M)$ by complex lines of \mathbb{C}^N through a common point.

From $h_*R = 0$, we have

$$R(\gamma) = \gamma^2, \quad \nabla_R R = 0 \quad \text{and} \quad \alpha_f(R, R) = 0. \quad (14)$$

From $h_*S = 0$ for S orthogonal to R , we obtain

$$S(\gamma) = 0, \quad \nabla_S R = -\gamma S \quad \text{and} \quad \alpha_f(R, S) = 0. \quad (15)$$

The last equations in (14) and (15) yield $R \in \Delta$, hence $JR \in \Delta$. On the other hand,

$$\ell_*R = f_*JR + R(1/\gamma)f_*JR + (1/\gamma)f_*J\nabla_R R + (1/\gamma)\alpha_f(JR, R).$$

The first two terms cancel out since $R(1/\gamma) = -1$ from (14). The third term is zero by (14) and the last one vanishes because $R \in \Delta$. Thus $\ell_*R = 0$. If S is orthogonal to R , we obtain that

$$\ell_*S = f_*JS + S(1/\gamma)f_*JS + (1/\gamma)f_*J\nabla_S R + (1/\gamma)\alpha_f(JS, R).$$

By the second equation in (15), we have $J\nabla_S R = -\gamma JS$, hence the first and third terms cancel out. The second term is zero by (15) and thus $\ell_*S = 0$. Hence, the map ℓ is constant.

By the second equations in (14) and (15), the latter applied to $S = JR$, we have that the distribution L is totally geodesic. Since L belongs to Δ , it follows that the leaves of L are mapped by f and g into affine subspaces of \mathbb{R}^N , as wished. ■

Minimal real Kaehler cones of dimension $n = 4$ admit a complete description from Theorem 27 of [2]. Start with a substantial minimal surface $g: M^2 \rightarrow \mathbb{R}^N$, $N \geq 5$, such that its *ellipse of curvature*, defined by

$$E(x) = \{\alpha_g(X, X) : X \in T_x M \text{ and } \|X\| = 1\},$$

is everywhere a circle. These surfaces can be easily described in terms of the generalized Weierstrass parametrization.

Proposition 13. *The map $F: N^4 := TM \rightarrow \mathbb{R}^N$ given by*

$$F(p, v) = g_*(p)v$$

defines, at regular points, a minimal immersion with the vertical distribution Δ of N^4 as relative nullity distribution. The leaves of Δ pass through the origin, hence F is a cone. Moreover, the induced metric gives N^4 the structure of a Kaehler manifold. Conversely, any minimal real Kaehler 4-dimensional cone is locally given in this way.

The preceding discussion leads to the following example of a pair of conformal immersions with the same Gauss map.

Example 14. Let $f: M^n \rightarrow \mathbb{R}^N$ be a minimal real Kaehler cone and let f_θ be a member of its associated family. Consider an inversion \mathcal{I} with respect to a sphere centered at the vertex of f_θ , and set $g = \mathcal{I} \circ f_\theta$. Then g is conformal to f with the same Gauss map.

5.3 The warped product examples

Our next examples require the notion of a warped product of isometric immersions introduced by Nölker [13].

5.3.1 Warped product of isometric immersions

Let $\mathbb{R}^N = \oplus_{i=0}^k V_i$ be an orthogonal decomposition into nontrivial subspaces, and let $z_1, \dots, z_k \in V_0$ satisfy $\langle z_i, z_j \rangle = 0$ for $1 \leq i \neq j \leq k$. For a fixed point $\bar{p} \in \mathbb{R}^N$, let S_i , $1 \leq i \leq k$, be the unique sphere or affine subspace of \mathbb{R}^N such that $V_i = T_{\bar{p}} S_i$ and whose mean curvature vector at \bar{p} is $-z_i$. Set $k_i = |z_i|^2$ and

$$S_0 = \bar{p} - \sum_{k_i > 0} k_i^{-1} z_i + \{p \in V_0 : \langle z_i, p \rangle > 0 \text{ for all } i \text{ with } k_i > 0\}.$$

Define $\sigma_i: S_0 \rightarrow \mathbb{R}_+$ by $\sigma_i(p) = 1 + \langle z_i, p - \bar{p} \rangle$ for $1 \leq i \leq k$ and

$$U = \mathbb{R}^N \setminus (\bar{p} - \sum_{k_i > 0} k_i^{-1} z_i + \cup_{k_i > 0} (\mathbb{R} z_i \oplus V_i)^\perp).$$

Then, the map $\Phi: S_0 \times_\sigma \prod_{i=1}^k S_i \rightarrow U$ given by

$$\Phi(p_0, \dots, p_k) = p_0 + \sum_{i=1}^k \sigma_i(p_0)(p_i - \bar{p})$$

is an isometry, called the *warped product representation* of \mathbb{R}^N determined by the data $(\bar{p}, S_1, \dots, S_k)$. Moreover, it was proved by Nölker [13] that any isometry of a warped product onto an open subset of \mathbb{R}^N is essentially given as the restriction of such a warped product representation.

Given immersions $f_i: M_i \rightarrow S_i$, $1 \leq i \leq k$, the map

$$f := \Phi \circ (f_0 \times \dots \times f_k): M := \prod_{i=0}^k M_i \rightarrow \mathbb{R}^N$$

is also an immersion whose induced metric is the warped product of the metrics induced by f_0, \dots, f_k , with warping function $\rho = (\rho_1, \dots, \rho_k)$ given by $\rho_i = \sigma_i \circ f_0$, $1 \leq i \leq k$. It is called the *warped product of f_0, \dots, f_k* .

In the following we only deal with the cases $k = 2, 3$. For $k = 2$, we take for simplicity the warped product representation $\Psi: \mathbb{R}_+^m \times \mathbb{S}^{N-m} \rightarrow \mathbb{R}^N$ given by

$$(X, Y) \mapsto (x_1, \dots, x_{m-1}, x_m Y), \quad (16)$$

whose induced metric is $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbb{R}^m} + x_m^2 \langle \cdot, \cdot \rangle_{\mathbb{S}^{N-m}}$.

For $k = 3$, we take $\Psi: \mathbb{R}_*^m \times \mathbb{S}^{m_1} \times \mathbb{S}^{m_2} \rightarrow \mathbb{R}^N$, $m_1 + m_2 = N - m$, given by

$$(X, Y) \mapsto (x_1, \dots, x_{m-2}, x_{m-1} Y_1, x_m Y_2), \quad (17)$$

for $X \in \mathbb{R}_*^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_{m-1}, x_m > 0\}$ and $Y_i \in \mathbb{S}^{m_i}$ for $1 \leq i \leq 2$.

5.3.2 Ordinary warped product examples

Let $f_0, g_0: N^s \rightarrow \mathbb{R}_+^m$ be immersions, let $\Psi: \mathbb{R}_+^m \times_{x_m} \mathbb{S}^{N-m} \rightarrow \mathbb{R}^N$ be the warped product representation (16) and let $\ell: S^{n-s} \rightarrow \mathbb{S}^{N-m}$ be an isometric immersion. We define $M^n = N^s \times S^{n-s}$ and $f, g: M^n \rightarrow \mathbb{R}^N$ by

$$f = \Psi \circ (f_0 \times \ell) \text{ and } g = \Psi \circ (g_0 \times \ell). \quad (18)$$

One can check that f, g have the same Gauss map if and only if f_0, g_0 do.

On the other hand, the metrics $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^\sim$ induced by f and g are, respectively,

$$\langle \cdot, \cdot \rangle_0 + \rho^2 \langle \cdot, \cdot \rangle_1 \text{ and } \langle \cdot, \cdot \rangle_0^\sim + \tilde{\rho}^2 \langle \cdot, \cdot \rangle_1,$$

where $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle_0^\sim$ are the metrics on N^s induced by f_0 and g_0 , respectively, $\langle \cdot, \cdot \rangle_1$ is the metric on S^{n-s} induced by ℓ , and $\rho = x_m \circ f_0$, $\tilde{\rho} = x_m \circ g_0$ are the last coordinate functions of f_0 and g_0 , respectively. Then, it is easily seen that $\langle \cdot, \cdot \rangle^\sim = \psi^2 \langle \cdot, \cdot \rangle$ for some $\psi \in C^\infty(M)$ if and only if

- (i) $\psi = \psi_0 \circ \pi_0$ for some $\psi_0 \in C^\infty(N)$,
- (ii) $\langle \cdot, \cdot \rangle_0^\sim = \psi_0^2 \langle \cdot, \cdot \rangle_0$,
- (iii) $\psi_0^2 \rho^2 = \tilde{\rho}^2$.

In other words, $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^\sim$ are conformal if and only if

$$\frac{1}{\tilde{\rho}^2} \langle \cdot, \cdot \rangle_0^\sim = \frac{1}{\rho^2} \langle \cdot, \cdot \rangle_0,$$

that is, f_0 and g_0 must induce the same metric from the hyperbolic metric on \mathbb{R}_+^m , in which case the conformal factor relating $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^\sim$ is $\psi = \psi_0 \circ \pi_0$, with $\psi_0^2 \rho^2 = \tilde{\rho}^2$.

Summarizing, we have the following fact.

Proposition 15. *The immersions f and g given by (18) are conformal with the same Gauss map if and only if f_0 and g_0 have the same Gauss map and induce the same metric from the hyperbolic metric on \mathbb{R}_+^m .*

Let $f_0 = \alpha: I \rightarrow \mathbb{R}_+^m$ and $g_0 = \beta: I \rightarrow \mathbb{R}_+^m$ be regular curves. Then f_0 and g_0 having the same Gauss map means that there exists $\lambda \in C^\infty(I)$ such that $\beta'(s) = \lambda(s)\alpha'(s)$ for all $s \in I$, whereas f_0 and g_0 inducing the same metric from the hyperbolic metric on \mathbb{R}_+^m should be understood as saying that α and β admit common unit-speed parametrizations as curves in the half-space model of hyperbolic space, i.e.,

$$\frac{|\beta'(s)|}{\beta_m(s)} = \frac{|\alpha'(s)|}{\alpha_m(s)}.$$

Therefore, either $\lambda(s) = \beta_m(s)/\alpha_m(s)$ or $\lambda(s) = -\beta_m(s)/\alpha_m(s)$. One can easily check that the first possibility leads to the trivial solution $\beta = C\alpha + v$ for some constant $C > 0$ and $v \in \mathbb{R}^m$. In the second one, from

$$\frac{\beta'_m(s)}{\beta_m(s)} = -\frac{\alpha'_m(s)}{\alpha_m(s)}$$

it follows that $\beta_m = C/\alpha_m$ for some constant $C > 0$. Thus $\lambda = -C/\alpha_m^2$, and hence

$$\beta = -C \int \frac{\alpha'(\tau)}{\alpha_m^2(\tau)} d\tau. \quad (19)$$

We have proved the following result.

Proposition 16. *Let $\alpha, \beta: I \rightarrow \mathbb{R}_+^m$ be regular curves, let $\Psi: \mathbb{R}_+^m \times_{x_m} \mathbb{S}^{N-m} \rightarrow \mathbb{R}^N$ be a warped product representation and let $\ell: S^{n-1} \rightarrow \mathbb{S}^{N-m}$ be an isometric immersion. Then $f, g: M^n \rightarrow \mathbb{R}^N$ defined by (18) for $M^n = I \times S^{n-1}$, are conformal immersions with the same Gauss map if and only if α and β are related by (19).*

Remark 17. If $m = 1$ and $\alpha, \beta: I \rightarrow \mathbb{R}$ are related by (19), then $\beta = C/\alpha$. In this case, $f, g: M^n \rightarrow \mathbb{R}^N$ given by (18) for $M^n = I \times S^{n-1}$ are cones that differ by an inversion with respect to a sphere centered at their common vertex.

By putting together Propositions 8 and 15 we get more interesting examples.

Proposition 18. *Let $f_0, g_0: N^2 \rightarrow \mathbb{R}_+^m$ be minimal surfaces, let $\Psi: \mathbb{R}_+^m \times_{x_m} \mathbb{S}^{N-m} \rightarrow \mathbb{R}^N$ be a warped product representation and let $\ell: S^{n-2} \rightarrow \mathbb{S}^{N-m}$ be any isometric immersion. Then $f, g: M^n \rightarrow \mathbb{R}^N$ given by (18), for $M^n = N^2 \times S^{n-2}$, are conformal immersions with the same Gauss map if and only if f_0 and g_0 are given as in Proposition 8.*

Remark 19. When f_0 (and hence also g_0) is totally geodesic, then $f(M)$ is either (an open subset of) a cylinder over ℓ or a product of a line with a cone over ℓ , according as $f_0(L^2)$ is parallel to the boundary \mathbb{R}^{m-1} of \mathbb{R}_+^m or not. Moreover, up to a translation we have $g(M) = f(M)$: the leaves of the product foliation of M^n corresponding to the first factor are relative nullity leaves of both f and g , and $g = f \circ \Phi$ for the conformal diffeomorphism of M^n given by $\Phi(x, y) = (h(x), y)$.

5.3.3 A triply warped product example

Start now with minimal surfaces $f_0, g_0: N^2 \rightarrow \mathbb{R}_*^m$. Let $\Psi: \mathbb{R}_+^m \times_{x_{m-1}} \mathbb{S}^{m_1} \times_{x_m} \mathbb{S}^{m_2} \rightarrow \mathbb{R}^N$ be the warped product representation (17) with $m_1 + m_2 = N - m$, and let $\ell_i: S^{s_i} \rightarrow \mathbb{S}^{m_i}$, $1 \leq i \leq 2$, be any isometric immersions, with $s_1 + s_2 = 2$. Set $M^n = N^2 \times S^{s_1} \times S^{s_2}$ and define $f, g: M^n \rightarrow \mathbb{R}^N$ by

$$f = \Psi \circ (f_0 \times \ell_1 \times \ell_2) \text{ and } g = \Psi \circ (g_0 \times \ell_1 \times (-\ell_2)). \quad (20)$$

The metrics $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^\sim$ induced by f and g are, respectively,

$$\langle \cdot, \cdot \rangle_0 + \sum_{i=1}^2 \rho_i^2 \langle \cdot, \cdot \rangle_i \text{ and } \langle \cdot, \cdot \rangle_0^\sim + \sum_{i=1}^2 \tilde{\rho}_i^2 \langle \cdot, \cdot \rangle_i,$$

where $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle_0^\sim$ are the metrics on N^2 induced by f_0 and g_0 , respectively, $\langle \cdot, \cdot \rangle_i$ is the metric on S^{s_i} induced by ℓ_i and $\rho_i = x_{m-2+i} \circ f_0$, $\tilde{\rho}_i = x_{m-2+i} \circ g_0$ are the two last coordinate functions of f_0 and g_0 , respectively.

Proposition 20. *The immersions f, g given by (20) induce conformal metrics on M^n and have the same Gauss map if and only if f_0, g_0 satisfy the following conditions:*

- (i) *If f_0 is parametrized in isothermal coordinates by $f_0 = (a_1, \dots, a_{m-2}, a, \bar{a})$ with $a, \bar{a} > 0$, then the function $A = a + i\bar{a}$ is holomorphic. Moreover, the coordinates are also isothermal for g_0 , and if $g_0 = (\alpha_1, \dots, \alpha_{m-2}, \alpha, \bar{\alpha})$ with $\alpha, \bar{\alpha} > 0$, then the function $\mathcal{A} = \alpha + i\bar{\alpha}$ is holomorphic.*
- (ii) *If \mathcal{R} denotes the reflection with respect to the hyperplane orthogonal to e_m , then $\mathcal{R} \circ g_0 = (\alpha_1, \dots, \alpha_{m-1}, \alpha, -\bar{\alpha})$ is related to f_0 by (11) and, in particular, $\mathcal{A} = 1/A$.*

Proof: It is easily seen that f, g have the same Gauss map if and only if $f_0, \mathcal{R} \circ g_0$ do. Moreover, $\langle \cdot, \cdot \rangle^\sim = \psi^2 \langle \cdot, \cdot \rangle$ for some $\psi \in C^\infty(M)$ if and only if

- (i) $\psi = \psi_0 \circ \pi_0$ for some $\psi_0 \in C^\infty(N)$,
- (ii) $\langle \cdot, \cdot \rangle_0^\sim = \psi_0^2 \langle \cdot, \cdot \rangle_0$,
- (iii) $\psi_0^2 \rho_i^2 = \tilde{\rho}_i^2$ for $1 \leq i \leq 2$.

Therefore, if f_0 is parametrized in isothermal coordinates (u, v) by $f_0 = (a_1, \dots, a_m)$, then (u, v) are also isothermal coordinates for g_0 , and

$$(\mathcal{R} \circ g_0)_z = e^\psi (f_0)_z \quad (21)$$

for some holomorphic function $\psi = \varphi + i\theta$, where $z = u + iv$. Let us denote temporarily $a = a_{m-1}$, $b = a_m$, $\alpha = \alpha_{m-1}$ and $\beta = \alpha_m$. Then, we have from (iii) and (21) that, one

one hand, $\alpha_z = e^\psi a_z, e^\varphi a = \alpha$ and, on the other hand, $\beta_z = -e^\psi b_z, e^\varphi b = \beta$. The first pair of equations leads, as in the proof of Proposition 8, to $e^\psi = 1/A^2$, where $A = a + i\bar{a}$ is holomorphic. A similar computation using the second pair gives $e^\psi = -1/B^2$, where $B = b + i\bar{b}$ is holomorphic. Therefore $A^2 = -B^2$, which implies that $b = \bar{a}$, and all of the remaining assertions follow. ■

Remark 21. A minimal surface S in \mathbb{R}^N having a pair of non-constant conjugate harmonic functions as coordinate functions is called 2-decomposable in [10]. Thus, there exists a direct sum decomposition of \mathbb{R}^N with respect to which S becomes the direct sum of a non-constant holomorphic function and a minimal surface in \mathbb{R}^{N-2} . By Proposition 4.1 of [10], this condition is equivalent to S being degenerate in the sense that its image by the \mathbb{Q}^{N-2} -valued Gauss map lies in a tangent hyperplane of the quadric \mathbb{Q}^{N-2} in \mathbb{CP}^{N-1} . In particular, if $N = 4$ then S must be a holomorphic curve.

6 The main result

We are now in a position to state our main result, namely, the classification of all pairs of conformal immersions into Euclidean space with the same Gauss map. We exclude the trivial case of Example 10 as well as the surface case discussed in Section 3.

Theorem 22. *Any pair $f, g: M^n \rightarrow \mathbb{R}^N$, $n \geq 3$, of immersions with the same map that are conformal but not isometric is as in Example 14 or Propositions 16, 18 or 20.*

For the proof of Theorem 22, we assume that we have a global (orthogonal) splitting

$$TM \otimes \mathbb{C} = L_+ \oplus L_- \oplus L_c$$

and make a case-by-case study according to the various possibilities for their ranks. After that, it is easy to see that solutions corresponding to different possibilities can not be glued together.

Before going into such study, we introduce the tools that are needed in order to show that a given isometric immersion into Euclidean space is a warped product of isometric immersions.

6.1 Hiepko and Nölker theorems

The first step is to show that the submanifold is intrinsically a warped product of Riemannian manifolds. This is accomplished by Hiepko's theorem stated below.

Recall that a subbundle E of the tangent bundle of a Riemannian manifold M is *umbilical* if there exists a section η of E^\perp , called the *mean curvature normal* of E , such that

$$\langle \nabla_X Y, Z \rangle = \langle X, Y \rangle \langle \eta, Z \rangle \quad \text{for all } X, Y \in E, Z \in E^\perp.$$

If, in addition,

$$\langle \nabla_X \eta, Z \rangle = 0, \text{ for all } X \in E, Z \in E^\perp,$$

then E is said to be a *spherical* subbundle.

In showing that a subbundle is spherical the following fact will be useful (cf. [16]).

Proposition 23. *Assume that E is an umbilical subbundle of TM of rank $E \geq 2$. If*

$$R(X, Y)Z \in E \text{ for all } X, Y, Z \in E,$$

then E is spherical. Moreover, the above condition holds if $f: M \rightarrow \mathbb{R}^N$ is an isometric immersion and $\alpha(E, E^\perp) = 0$.

Proof: By assumption, there exists a vector field $\eta \in E^\perp$ such that

$$(\nabla_X Y)_{E^\perp} = \langle X, Y \rangle \eta \text{ for all } X, Y \in E.$$

We must show that

$$\langle \nabla_Y \eta, Z \rangle = 0 \text{ for all } Y \in E, Z \in E^\perp. \quad (22)$$

For an orthonormal pair $X, Y \in E$, we have

$$\langle \nabla_Y \eta, Z \rangle = Y \langle \nabla_X X, Z \rangle - \langle \eta, \nabla_Z Y \rangle = \langle \nabla_Y \nabla_X X, Z \rangle + \langle \nabla_X X, \nabla_Y Z \rangle - \langle \eta, \nabla_Z Y \rangle$$

Using $\langle R(Y, X)X, Z \rangle = 0$, we obtain

$$\langle \nabla_Y \nabla_X X, Z \rangle = \langle \nabla_X \nabla_Y X, Z \rangle + \langle \nabla_{[Y, X]} X, Z \rangle = \langle [Y, X], X \rangle \langle \eta, Z \rangle = \langle \nabla_X X, Y \rangle \langle \eta, Z \rangle.$$

On the other hand,

$$\begin{aligned} \langle \nabla_X X, \nabla_Y Z \rangle &= \langle \eta, \nabla_Y Z \rangle + \langle \nabla_X X, (\nabla_Y Z)_{L_+} \rangle = \langle \eta, \nabla_Y Z \rangle + \langle \nabla_X X, Y \rangle \langle Y, \nabla_Y Z \rangle \\ &= \langle \eta, \nabla_Y Z \rangle - \langle \nabla_X X, Y \rangle \langle \eta, Z \rangle, \end{aligned}$$

and (22) follows. For the last assertion, the Gauss equation and the assumption give

$$R(X, Y)U = A_{\alpha(Y, U)} - A_{\alpha(X, U)} = 0 \text{ for all } U \in E^\perp. \quad \blacksquare$$

We can now state Hiepko's [9] theorem.

Theorem 24. *Let M^n be a Riemannian manifold and let $TM = L \oplus S_1 \oplus \cdots \oplus S_k$ be an orthogonal decomposition into nontrivial vector subbundles such that S_1, \dots, S_k are spherical and $S_1^\perp, \dots, S_k^\perp$ totally geodesic. Then, there is locally a decomposition of M^n into a Riemannian warped product $M^n = N_0 \times_{\varrho_1} N_1 \times \cdots \times_{\varrho_k} N_k$ such that $L = TN_0$ and $S_i = TN_i$ for $1 \leq i \leq k$.*

The problem of determining whether an isometric immersion of a warped product manifold is a warped product of isometric immersions of the factors is handled by the following result of Nölker [13].

Theorem 25. *Let $f: M^n \rightarrow \mathbb{R}^N$ be an isometric immersion of a warped product manifold $M^n = M_0 \times_{\varrho_1} M_1 \times \cdots \times_{\varrho_k} M_k$ whose second fundamental form satisfies*

$$\alpha(X_i, X_j) = 0 \quad \text{for all } X_i \in TM_i, X_j \in TM_j, \quad i \neq j.$$

Given $\bar{p} = (\bar{p}_0, \dots, \bar{p}_k) \in M^n$, set $f_i = f \circ \tau_i^{\bar{p}}: M_i \rightarrow \mathbb{R}^N$ for $\tau_i^{\bar{p}}(p_i) = (\bar{p}_0, \dots, p_i, \dots, \bar{p}_k)$, and let S_i be the spherical hull of f_i , $1 \leq i \leq k$. Then f_0 is an isometric immersion, f_i is a homothetical immersion with homothety factor $\rho_i(\bar{p}_0)$ and $(f(\bar{p}); S_1, \dots, S_k)$ determines a warped product representation $\Phi: S_0 \times_{\sigma_1} S_1 \times \cdots \times_{\sigma_k} S_k \rightarrow \mathbb{R}^N$ such that $f_0(M_0) \subset S_0$, $\rho_i = \rho_i(\bar{p}_0)(\sigma_i \circ f_0)$ and

$$f = \Phi \circ (f_0 \times \cdots \times f_k),$$

where f_i is regarded as a map into S_i for $1 \leq i \leq k$.

6.2 Basic lemmata

Next, we derive some basic lemmata to be used throughout the proof of our main result. By assumption, the conformal factor e^φ relating the metrics induced by f and g satisfies $\nabla\varphi \neq 0$ on any open subset. Therefore, from now on we assume that $\nabla\varphi \neq 0$ everywhere without loss of generality.

Lemma 26. *The following facts hold:*

(i) *The subbundle L_+ is umbilical with mean curvature vector η_+ given by*

$$(T - I)\eta_+ = T(\nabla\varphi)_{L_+^\perp}, \quad (23)$$

(ii) *If $\text{rank } L_+ \geq 2$, then $\nabla\varphi \in L_+^\perp$ and L_+ is spherical.*

Proof: Applying (2) for $Y \in L_+$ gives

$$(T - I)\nabla_X Y = \langle X, Y \rangle T\nabla\varphi - Y(\varphi)TX. \quad (24)$$

Since the left-hand-side belongs to L_+^\perp , then the same holds for the other side. If $\text{rank } L_+ \geq 2$, choosing $0 \neq X \in L_+$ orthogonal to Y yields $\nabla\varphi \in L_+^\perp$ and $(\nabla_X Y)_{L_+^\perp} = 0$. Hence, L_+ is umbilical and its mean curvature vector field η_+ satisfies $(T - I)\eta_+ = T\nabla\varphi$. If $\text{rank } L_+ = 1$, then (23) follows by applying (24) to a unit vector field $X = Y \in L_+$. The last assertion in (ii) is a consequence of Proposition 23. ■

Lemma 27. *Let L be any of the vector subbundles L_+ , L_c , L_+^\perp or L_c^\perp . Then L is totally geodesic if and only if $\nabla\varphi \in L$.*

Proof: For the “only if” part observe that if $X \in L$ is any unit vector field, then we have from (2) that

$$\nabla\varphi = \nabla_X X - T^t \nabla_X T X + X(\varphi)X \in L.$$

We now prove the converse. That L_+ is totally geodesic if $\nabla\varphi \in L_+$ follows from Lemma 26-(i). We also have from this result that $(\eta_+)_{L_c} = 0 = (\eta_-)_{L_c}$ if $\nabla\varphi \in L_c^\perp$. On the other hand, regardless of $\nabla\varphi \in L_c^\perp$, we obtain from (24) that $\nabla_X Y, \nabla_Y X \in L_c^\perp$ for $X \in L_-$ and $Y \in L_+$. Hence, L_c^\perp is totally geodesic if $\nabla\varphi \in L_c^\perp$. If $\nabla\varphi \in L_+^\perp$, applying (24) for $X \in L_+^\perp$ implies that $\nabla_X Y \in L_+$ for any $Y \in L_+$. Thus L_+^\perp is totally geodesic. Similarly for L_-^\perp . It follows that $L_c = L_+^\perp \cap L_-^\perp$ is totally geodesic if $\nabla\varphi \in L_c$. ■

Lemma 28. *Assume $L_c \neq 0$. Then the following facts hold:*

(i) $\nabla\varphi \notin L_+$,

(ii) If $\text{rank } L_+ = 1$ and $L_+ \not\subset \Delta$, then $\nabla\varphi \in L_+^\perp$.

Proof: (i) If $\text{rank } L_+ \geq 2$, the assertion follows from Lemma 26-(ii) since $\nabla\varphi \neq 0$. Assume $\text{rank } L_+ = 1$. The inner product of (6) with $X \in L_+$ for $\mu = \bar{\lambda}$ and $W = \bar{Z}$ gives

$$(1 - \bar{\lambda})\langle \nabla_Z \bar{Z}, X \rangle = \langle Z, \bar{Z} \rangle X(\varphi) \quad \text{for any } Z \in E_\lambda. \quad (25)$$

If $\nabla\varphi$ spans L_+ , then L_+^\perp is integrable. Thus $\langle \nabla_Z \bar{Z}, X \rangle$ is real which contradicts (25).

(ii) Applying (6) for $\mu = \lambda$ and $W = Z$ yields

$$Z(\lambda)Z + (\lambda I - T)\nabla_Z Z = \lambda Z(\varphi)Z. \quad (26)$$

Taking the inner product with $X \in L_+$ gives

$$\langle \nabla_Z Z, X \rangle = 0. \quad (27)$$

On the other hand, the Codazzi equation yields

$$\alpha(\nabla_Z X, \bar{Z}) + \alpha(X, \nabla_Z \bar{Z}) = \alpha(\nabla_X Z, \bar{Z}) + \alpha(Z, \nabla_X \bar{Z}) = 0.$$

We obtain using (27) that

$$\langle \nabla_Z \bar{Z}, X \rangle \alpha(X, X) = 0.$$

If $\alpha(X, X) \neq 0$, it follows that $\langle \nabla_Z \bar{Z}, X \rangle = 0$, hence $X(\varphi) = 0$ by (25). ■

Lemma 29. *The following facts hold:*

(i) A complex eigenvalue λ of T is constant on $E_\lambda \oplus E_{\bar{\lambda}}$ if and only if $\nabla\varphi \in (E_\lambda \oplus E_{\bar{\lambda}})^\perp$,

(ii) If $\text{rank } L_c \geq 4$, then a complex eigenvalue μ of T can only fail to be constant along $E_\mu \oplus E_{\bar{\mu}}$. Moreover, this may only happen if μ is simple and $E_\mu \oplus E_{\bar{\mu}} = \Delta \otimes \mathbb{C}$,

(iii) If $\text{rank } L_c \geq 4$ and either $\Delta \cap L_c = \{0\}$ or $\text{rank } \Delta \geq 3$, then $\nabla\varphi \in L_c^\perp$,

(iv) If $\text{rank } L_c = 2$, $\text{rank } \Delta \geq 3$ and $L_c \subset \Delta$, then $\nabla\varphi \in L_c^\perp$.

Proof: (i) Let $Z \in E_\lambda$. We have from (26) that

$$\bar{\lambda}Z(\lambda) = Z(\varphi). \quad (28)$$

Hence $\bar{Z}(\varphi) = \lambda\bar{Z}(\bar{\lambda}) = -\bar{\lambda}\bar{Z}(\lambda)$, and the assertion follows.

(ii) Applying (7) to $W \in E_\mu$ gives

$$X(\mu) = 0 = Y(\mu) \quad \text{for } X \in L_+, Y \in L_-. \quad (29)$$

Let $Z \in E_\lambda$ and assume that $\langle Z, W \rangle = 0$ if $\lambda = \bar{\mu}$. Then (6) yields $Z(\mu) = 0$. This and (29) show that μ can only fail to be constant along $E_\mu \oplus E_{\bar{\mu}}$. Moreover, this can only happen if μ is simple, since $\text{rank } L_c \geq 4$.

We now show that μ is also constant along $E_\mu \oplus E_{\bar{\mu}}$ unless (μ is simple and) $E_\mu \oplus E_{\bar{\mu}} \subset \Delta \otimes \mathbb{C}$. Choose a complex eigenvalue $\lambda \notin \{\mu, \bar{\mu}\}$. By the Codazzi equation

$$\alpha(\nabla_Z \bar{Z}, W) + \alpha(\bar{Z}, \nabla_Z W) = \alpha(\nabla_{\bar{Z}} Z, W) + \alpha(Z, \nabla_{\bar{Z}} W)$$

for any $Z \in E_\lambda$. Using that $\alpha(Z, \bar{Z}) = 0$, we obtain

$$\langle \nabla_Z \bar{Z}, \bar{W} \rangle \alpha(W, W) - \langle \nabla_Z Z, W \rangle \alpha(\bar{Z}, \bar{Z}) = \langle \nabla_{\bar{Z}} Z, \bar{W} \rangle \alpha(W, W) - \langle \nabla_{\bar{Z}} \bar{Z}, W \rangle \alpha(Z, Z). \quad (30)$$

On the other hand, it follows from (6) that

$$\langle \nabla_Z Z, W \rangle = 0 = \langle \nabla_{\bar{Z}} \bar{Z}, W \rangle. \quad (31)$$

By (30) and (31) we have

$$\langle [Z, \bar{Z}], \bar{W} \rangle \alpha(W, W) = 0 = \langle [Z, \bar{Z}], W \rangle \alpha(\bar{W}, \bar{W}).$$

We obtain that

$$\alpha(W, W) = 0 \quad \text{or} \quad \langle [Z, \bar{Z}], \bar{W} \rangle = 0 = \langle [Z, \bar{Z}], W \rangle.$$

Assume $\alpha(W, W) \neq 0$. We also have from (6) that

$$(\lambda - \mu) \langle \nabla_Z \bar{Z}, W \rangle = \lambda W(\varphi) \quad \text{and} \quad (\bar{\lambda} - \mu) \langle \nabla_{\bar{Z}} Z, W \rangle = \bar{\lambda} W(\varphi). \quad (32)$$

If $\langle \nabla_Z \bar{Z}, W \rangle \neq 0$, we obtain that $\lambda = \bar{\lambda}$, a contradiction. Hence,

$$\langle \nabla_Z \bar{Z}, W \rangle = 0 = \langle \nabla_{\bar{Z}} Z, W \rangle.$$

Thus $\nabla\varphi \in (E_\mu \oplus E_{\bar{\mu}})^\perp$ by (32), and the conclusion follows from (i).

To complete the proof, it remains to show that μ is also constant along $E_\mu \oplus E_{\bar{\mu}}$ if $E_\mu \oplus E_{\bar{\mu}}$ is properly contained in $\Delta \otimes \mathbb{C}$. If this is the case, then Δ has rank at least three. Since T leaves invariant any leaf σ of Δ , we can apply Proposition 11 to $f|_\sigma$ and conclude that $e^\varphi T|_\Delta$ is the derivative of a conformal transformation ϕ of σ . By Liouville's theorem ϕ is a Moebius transformation, hence $e^\varphi T$ has constant eigenvalues along $\Delta \otimes \mathbb{C}$. This also gives (iv). Then (iii) is a consequence of (i) and (ii). ■

Let Δ be a totally geodesic distribution on a Riemannian manifold. The corresponding *splitting tensor* C associates to each $S \in \Delta$ the map $C_S: \Delta^\perp \rightarrow \Delta^\perp$ defined by

$$C_S X = -(\nabla_X S)_{\Delta^\perp}.$$

It is well-known that C satisfies the differential equation

$$\nabla_T C_S = C_S C_T + C_{\nabla_T S}$$

for all $S, T \in \Delta$ (cf. [3]).

Lemma 30. *If $TM \otimes \mathbb{C} = L_c$ then $n = 2$.*

Proof: By Lemma 29, it suffices to show that if $\text{rank } L_c \geq 4$, then there can not exist a simple complex eigenvalue μ of T such that $\Delta \otimes \mathbb{C} = E_\mu \oplus E_{\bar{\mu}}$. Assume otherwise, and let $0 \neq W \in E_\mu$ and $Z \in E_\lambda$ with $\lambda \neq \mu, \bar{\mu}$. Thus $\alpha(W, W) = 0$ and $\alpha(Z, Z) \neq 0$. We have from Lemma 29 that $\lambda = \alpha + i\beta$ is constant on M^n , that

$$\langle \nabla_Z Z, W \rangle = 0 = \langle \nabla_Z Z, \bar{W} \rangle \quad (33)$$

and that

$$(1 - \bar{\lambda}\mu)\langle \nabla_Z \bar{Z}, W \rangle = (1 - \lambda\mu)\langle \nabla_{\bar{Z}} Z, W \rangle. \quad (34)$$

Take an orthonormal frame X, Y of Δ that is constant along each leaf. It follows from (33) that the complexified splitting tensor C of Δ satisfies

$$C_X Z = -\nabla_Z X = \langle Z, \bar{Z} \rangle^{-1} \langle \nabla_Z \bar{Z}, X \rangle Z \quad \text{for all } Z \in E_\lambda.$$

Set $S = \langle Z, \bar{Z} \rangle^{-1} \nabla_Z \bar{Z}$ and $\rho = \langle S, X \rangle$. Then $C_X Z = \rho Z$, and similarly $C_Y Z = \nu Z$, with $\nu = \langle S, Y \rangle$. Writing $S = U + iV$, we obtain from (34) that

$$\mu = \frac{\langle V, W \rangle}{\langle \alpha V - \beta U, W \rangle}.$$

Since $\mu\bar{\mu} = 1$, we have $\|V\|^2 = \|\alpha V - \beta U\|^2$. Since $\beta \neq 0$, this can also be written as

$$\beta(\|U\|^2 - \|V\|^2) = 2\alpha\langle V, U \rangle. \quad (35)$$

Denote $A = \langle S, S \rangle = (\|U\|^2 - \|V\|^2) + 2i\langle U, V \rangle$. Then (35) implies that A/\bar{A} is constant on M^n . Thus,

$$X(A)\bar{A} = AX(\bar{A}). \quad (36)$$

On the other hand, from $\nabla_X C_X = C_X^2$ and $\nabla_X C_Y = C_Y C_X$ applied to $Z \in E_\lambda$ we obtain $X(\rho) = \rho^2$ and $X(\nu) = \nu\rho$. Since $A = \rho^2 + \nu^2$, this gives $X(A) = 2\rho A$. Replacing into (36) yields $(\rho - \bar{\rho})|A|^2 = 0$. Thus $\rho = \bar{\rho}$, that is, $\langle V, X \rangle = 0$. Similarly, $\langle V, Y \rangle = 0$. Hence $V = 0$, a contradiction. ■

Lemma 31. *If L_c is totally geodesic, then $\text{rank } L_c = 2$ and L_+ is spherical.*

Proof: Since L_c is T -invariant, the first assertion follows by applying Lemma 30 to the restriction of f to a leaf of L_c . For the second assertion, by Lemma 26-(ii) we may assume that $\text{rank } L_+ = 1$. Since $\nabla\varphi \in L_c$ by Lemma 27, applying (5) to unit vector fields $X \in L_+$ and $Y \in L_-$ yields $2\langle \eta_+, Y \rangle = Y(\varphi) = 0$, and hence $\eta_+ \in L_c$. Let $\lambda, \bar{\lambda}$ be the complex eigenvalues of T , let $Z \in E_\lambda$ and let X be a unit vector field spanning L_+ . We obtain from (7) that $X(\lambda) = 0$ and $Z(\varphi) = \langle \eta_+, Z \rangle(1 - \lambda)$. Also $Y(\lambda) = 0$ if $Y \in L_-$. Hence, $\nabla\lambda \in L_c$. From (28) we have $Z(\varphi) = \bar{\lambda}Z(\lambda)$. Thus,

$$(1 - \lambda)\langle \eta_+, Z \rangle = \bar{\lambda}Z(\lambda).$$

Taking the X derivative and then using $\nabla\lambda \in L_c$ and that L_c is totally geodesic yield

$$\begin{aligned} (1 - \lambda)\langle \nabla_X \eta_+, Z \rangle + (1 - \lambda)\langle \eta_+, \nabla_X Z \rangle &= \bar{\lambda}XZ(\lambda) = \bar{\lambda}[X, Z](\lambda) = \bar{\lambda}\nabla_X Z(\lambda) \\ &= \bar{\lambda}\langle \nabla_X Z, \bar{Z} \rangle Z(\lambda) = (1 - \lambda)\langle \nabla_X Z, \bar{Z} \rangle \langle \eta_+, Z \rangle = (1 - \lambda)\langle \eta_+, \nabla_X Z \rangle, \end{aligned}$$

and therefore $\langle \nabla_X \eta_+, Z \rangle = 0$. ■

7 The proof of Theorem 22

We now prove Theorem 22 through a case-by-case study of the orthogonal splitting $TM \otimes \mathbb{C} = L_+ \oplus L_- \oplus L_c$ of the complexified tangent bundle of a submanifold $f: M^n \rightarrow \mathbb{R}^N$ that admits a conformal deformation $g: M^n \rightarrow \mathbb{R}^N$ with the same Gauss map.

In the following, we always assume that $n \geq 3$. We also suppose that f and g are neither totally geodesic nor differ by a homothety and a translation.

7.1 The case $L_c = \{0\}$.

We begin with the case in which L_c is trivial. In particular, the next lemma provides a simpler proof of Theorem 18 in [16].

Lemma 32. *If L_c is trivial, then f and g are as in Proposition 16.*

Proof: If $\text{rank } L_+, L_- \geq 2$, we obtain from Lemma 26-(ii) that $\nabla\varphi = 0$. Thus, we may assume $\text{rank } L_+ = 1$. Since $n \geq 3$, we have that $\text{rank } L_- \geq 2$. Thus L_- is spherical and $\nabla\varphi \in L_+$ by Lemma 26-(ii), and hence L_+ is totally geodesic by Lemma 27. Bearing in mind Lemma 4-(ii), we obtain from Theorems 24 and 25 that f and g are given as in (18) for some regular curves $f_0 = \alpha: I \rightarrow \mathbb{R}_+^m$ and $g_0 = \beta: I \rightarrow \mathbb{R}_+^m$. The conclusion now follows from Proposition 16. ■

7.2 The case $L_- = \{0\}$.

The next case to consider is when either L_+ or L_- is trivial, since the case $TM \otimes \mathbb{C} = L_c$ has already been treated in Lemma 30. In view of Fact 6, there is no loss of generality in assuming that L_- is trivial.

Lemma 33. *If L_- is trivial, then f and g are as in Proposition 18.*

Proof: If $\text{rank } L_+ \geq 2$, then L_+ is spherical and $\nabla\varphi \in L_c$ by Lemma 26-(ii). Then L_c is totally geodesic by Lemma 27, and hence $\text{rank } L_c = 2$ by Lemma 31. In view of Lemma 4-(ii), we obtain from Theorems 24 and 25 that f and g are given as in (18) for minimal surfaces $f_0, g_0: N^2 \rightarrow \mathbb{R}_+^m$. The conclusion now follows from Proposition 18.

Assume $\text{rank } L_+ = 1$ and $L_+ \subset \Delta$. Since f and g are not totally geodesic, then either L_+ is the common relative nullity distribution or $\text{rank } L_c \geq 4$ and $\text{rank } \Delta \geq 3$. In any case, $\nabla\varphi \in L_+$ by Lemmas 27 and 29, a contradiction with Lemma 28-(i).

Suppose $\text{rank } L_+ = 1$ and $L_+ \not\subset \Delta$. Then L_c is totally geodesic by Lemma 28-(ii). Then $\text{rank } L_c = 2$ and L_+ is spherical by Lemma 31. The conclusion follows exactly as in the case of $\text{rank } L_+ \geq 2$. ■

7.3 The case $L_+ \neq \{0\}$, $L_- \neq \{0\}$ and $L_c \neq \{0\}$.

Finally, we treat the case in which L_+ , L_- and L_c are all assumed to be nontrivial.

Lemma 34. *If either*

- (i) *rank $L_+ \geq 2$ and rank $L_- \geq 2$, or*
- (ii) *rank $L_+ = 1$, $L_+ \not\subset \Delta$ and rank $L_- \geq 2$, or*
- (iii) *rank $L_+ = 1 = \text{rank } L_-$ and $L_+, L_- \not\subset \Delta$,*

then f and g are as in Proposition 20.

Proof: We will prove that either one of assumptions (i), (ii) or (iii) implies that both L_+ and L_- are spherical and that $\nabla\varphi \in L_c$. Then L_c is totally geodesic by Lemma 27, and hence $\text{rank } L_c = 2$ by Lemma 31. As before, the conclusion follows from Lemma 4-(ii), Theorems 24 and 25 and Proposition 20.

If (i) holds, the proof follows from Lemma 26-(ii). If (ii) holds, we obtain from Lemma 26-(ii) that L_- is spherical and $\nabla\varphi \in L_-^\perp$. By Lemma 28-(ii) that $\text{rank } L_+ = 1$ and $L_+ \not\subset \Delta$ imply $\nabla\varphi \in L_+^\perp$. Thus $\nabla\varphi \in L_c$, and hence L_c is totally geodesic by Lemma 27. Then L_+ is spherical by Lemma 31. Under the assumptions in (iii), we have from Lemma 28-(ii) that $\nabla\varphi \in L_c$ and, as before, L_+ and L_- are spherical. ■

The next two lemmas take care of the remaining case.

Lemma 35. *If $\text{rank } L_+ = 1$ and $L_+ \subset \Delta$, then $\text{rank } L_- = 1$ and $L_+ \oplus L_- \subset \Delta$.*

Proof: Assume otherwise that either $\text{rank } L_- \geq 2$ or $\text{rank } L_- = 1$ and $L_- \not\subset \Delta$. Then $\nabla\varphi \in L_-^\perp$ by Lemma 26-(ii) or Lemma 28-(ii), respectively. If $\text{rank } L_c \geq 4$, then either $\Delta \cap L_c = \{0\}$ or $\text{rank } \Delta \geq 3$. In both cases, $\nabla\varphi \in L_c^\perp$ by Lemma 29, and hence $\nabla\varphi \in L_+$. If $\text{rank } L_c = 2$, then either $\text{rank } \Delta \geq 3$ and $L_c \subset \Delta$ or $L_c \cap \Delta = \{0\}$. In the first case, Lemma 29-(iv) implies that $\nabla\varphi \in L_c^\perp$, hence $\nabla\varphi \in L_+$. We reach the same conclusion in the second case, for now $\nabla\varphi \in L_-^\perp \cap \Delta = L_+$. But this is in contradiction with Lemma 28-(i). ■

Lemma 36. *If $\text{rank } L_+ = 1 = L_-$ and $L_+ \oplus L_- \subset \Delta$, then f, g are as in Example 14.*

For the convenience of the reader we divide the proof into three sublemmas.

Sublemma 37. *The following holds:*

- (i) *The subbundle $L := L_+ \oplus L_-$ is totally geodesic and $\nabla\varphi \in L$,*
- (ii) *T has only one pair of complex conjugate eigenvalues $\lambda = a + ib$ and $\bar{\lambda}$,*
- (iii) *There exists an orthonormal frame $\{R, S\}$ of L such that*

$$TR = -aR - bS \quad \text{and} \quad TS = -bR + aS, \quad (37)$$

$$C_R = \gamma I \quad \text{and} \quad C_S|_{E_\lambda} = i\gamma I, \quad (38)$$

$$\nabla_R R = 0 \quad \text{and} \quad \nabla_S S = \gamma R, \quad (39)$$

$$R(\gamma) = \gamma^2 \quad \text{and} \quad S(\gamma) = 0, \quad (40)$$

where C is the complexified splitting tensor of L .

Proof: (i) If $L = \Delta$, then $L = L_c^\perp$ is totally geodesic, thus $\nabla\varphi \in L$ by Lemma 27. Otherwise $\text{rank } \Delta \geq 3$, and we conclude again that $\nabla\varphi \in L$ from Lemma 29. Thus, also in this case we have that L is totally geodesic by Lemma 27.

(ii) Let X, Y be unit vector fields spanning L_+, L_- , respectively. From (23) and the similar formula for η_- we have

$$\frac{1}{2}\nabla\varphi = \nabla_X X + \nabla_Y Y = \Gamma_2 X + \Gamma_1 Y, \quad (41)$$

where $\Gamma_1 = \langle \nabla_X X, Y \rangle$ and $\Gamma_2 = \langle \nabla_Y Y, X \rangle$. Take $\sqrt{2}Z = u + iv \in E_\lambda$ with $\{u, v\}$ orthonormal. We obtain from (27) that

$$(\nabla_u u)_L = (\nabla_v v)_L \text{ and } (\nabla_u v + \nabla_v u)_L = 0 \quad (42)$$

Using (42) it follows from (25) and (41) that

$$(a-1)\langle \nabla_u u, X \rangle + b\langle \nabla_v u, X \rangle = -2\Gamma_2 \text{ and } (a-1)\langle \nabla_v u, X \rangle - b\langle \nabla_u u, X \rangle = 0.$$

Similarly,

$$(a+1)\langle \nabla_u u, Y \rangle + b\langle \nabla_v u, Y \rangle = 2\Gamma_1 \text{ and } (a+1)\langle \nabla_v u, Y \rangle - b\langle \nabla_u u, Y \rangle = 0.$$

It follows that

$$(\nabla_u u)_L = \frac{1}{2}\nabla\varphi \quad (43)$$

and

$$\langle \nabla_v u, X \rangle = \frac{b}{(a-1)}\Gamma_2, \quad \langle \nabla_v u, Y \rangle = \frac{b}{(a+1)}\Gamma_1. \quad (44)$$

Since $\langle \nabla_u \nabla\varphi, v \rangle = \langle \nabla_v \nabla\varphi, u \rangle$, using (42) and (43) we obtain

$$\langle (\nabla_u u)_L, (\nabla_v u)_L \rangle = 0.$$

From (43) and (44) we have

$$\langle \nabla_v u, X \rangle \langle \nabla_v u, Y \rangle = -\langle \nabla_u u, X \rangle \langle \nabla_u u, Y \rangle.$$

Hence,

$$\|(\nabla_u u)_L\| = \|(\nabla_v u)_L\|.$$

In view of (41), (43) and (44), we now have that $(a+1)\Gamma_2^2 + (a-1)\Gamma_1^2 = 0$. Hence,

$$a = \frac{\Gamma_1^2 - \Gamma_2^2}{\Gamma_1^2 + \Gamma_2^2} \quad \text{and} \quad b = \pm \frac{2\Gamma_1\Gamma_2}{\Gamma_1^2 + \Gamma_2^2}. \quad (45)$$

Thus, T has exactly one pair of complex conjugate eigenvalues.

(iii) Consider the orthonormal frame $\{R, S\}$ of L given by

$$R = (1/2\gamma)\nabla\varphi = (1/\gamma)(\Gamma_2 X + \Gamma_1 Y) \text{ and } S = (1/\gamma)(-\Gamma_1 X + \Gamma_2 Y)$$

where $2\gamma = |\nabla\varphi|$. From (42), (43) and (44) we have

$$(\nabla_u u)_L = \gamma R = (\nabla_v v)_L \text{ and } (\nabla_v u)_L = \gamma S = -(\nabla_u v)_L. \quad (46)$$

Using (45) and choosing b with the plus sign in (45) we obtain (37). On the other hand,

$$\langle \nabla_u \nabla\varphi, S \rangle = \langle \nabla_S \nabla\varphi, u \rangle = 0 \quad \text{and} \quad \langle \nabla_v \nabla\varphi, S \rangle = \langle \nabla_S \nabla\varphi, v \rangle = 0$$

yield

$$\langle \nabla_u R, S \rangle = 0 = \langle \nabla_v R, S \rangle. \quad (47)$$

From (6) we have for $Z \perp \{W, \bar{W}\}$ that

$$\langle \nabla_Z W, X \rangle = 0 = \langle \nabla_Z W, Y \rangle \quad \text{and} \quad \langle \nabla_Z \bar{W}, X \rangle = 0 = \langle \nabla_Z \bar{W}, Y \rangle. \quad (48)$$

It follows from (46), (47) and (48) that

$$\nabla_Z R = -\gamma Z \quad \text{and} \quad \nabla_Z S = -i\gamma Z, \quad \text{for all } Z \in E_\lambda, \quad (49)$$

which is equivalent to (38). Finally, from $\nabla_R C_R = C_R^2 + \alpha C_S$ and $\nabla_S C_S = C_S^2 + \beta C_R$ we obtain (39) and (40).

Sublemma 38. *Both f and g are cones.*

Proof: To prove that f is a cone we show that $h = f + \gamma^{-1} f_* R$ is a constant map. Using that $R \in \Delta$ we have

$$h_* R = f_* R + R(1/\gamma) f_* R + (1/\gamma) f_* \nabla_R R = 0$$

by (39) and the first equations in (40). Also,

$$h_* S = f_* S + S(1/\gamma) f_* R + (1/\gamma) f_* \nabla_S R = 0$$

by the second equation in (40) and $\nabla_S R = -\gamma S$, which follows from (39). Finally, since

$$2Z(\gamma^2) = \langle \nabla_Z \nabla \varphi, \nabla \varphi \rangle = \langle \nabla_{\nabla \varphi} \nabla \varphi, Z \rangle = 0$$

for $\nabla \varphi \in L$ and L is totally geodesic, we have $Z(\gamma) = 0$ for any $Z \in L_c$. It follows using (49) that

$$h_* Z = f_* Z + Z(1/\gamma) f_* R + (1/\gamma) f_* \nabla_Z R = 0.$$

Now set $\tilde{R} = e^{-\varphi} R$, $\tilde{\gamma} = e^{-\varphi} \gamma$ and $\tilde{S} = e^{-\varphi} S$. Using (3) we obtain

$$\tilde{\nabla}_Z \tilde{R} = \tilde{\gamma} Z, \quad \tilde{\nabla}_{\tilde{R}} \tilde{R} = 0 \quad \text{and} \quad \tilde{\nabla}_{\tilde{S}} \tilde{S} = -\tilde{\gamma} \tilde{R}.$$

Then, a computation similar to the above shows that $\ell = g - \tilde{\gamma}^{-1} g_* \tilde{R}$ is also constant.

Sublemma 39. *The manifold M^n is Kaehler, the immersion f is minimal and there exist an inversion \mathcal{I} and a member f_θ of the associated family of f such that $g = \mathcal{I} \circ f_\theta$ up to a homothety.*

Proof: For ℓ as in the end of the proof of Sublemma 38, let $Q_0 \in \mathbb{R}^N$ be its constant value and let \mathcal{I} be an inversion with respect to a unit sphere centered at Q_0 . Notice that \mathcal{I} leaves $g(M)$ invariant. The differential of \mathcal{I} at p is

$$\mathcal{I}_*(p) = \frac{1}{|p - Q_0|^2} \mathcal{R},$$

where \mathcal{R} is the reflection with respect to the hyperplane orthogonal to the position vector $p - Q_0$. Since $g(p) - Q_0 = \tilde{\gamma}^{-1} g_* \tilde{R}$, it follows that $|g(p) - Q_0|^2 = \tilde{\gamma}^{-2}$. Hence,

$$(\mathcal{I} \circ g)_* = \tilde{\gamma}^2 g_* \hat{\mathcal{R}} = \tilde{\gamma}^{-2} f_* e^\varphi \hat{\mathcal{R}} T = e^{-\varphi} \gamma^2 f_* \hat{T}$$

where $\hat{\mathcal{R}}$ is defined by $\hat{\mathcal{R}}R = -R$ and $\hat{\mathcal{R}}|_{R^\perp} = I|_{R^\perp}$, and $\hat{T} = \hat{\mathcal{R}}T$. Now, the first equation in (40) and $\nabla\varphi = 2\gamma R$ give $R(e^{-\varphi}\gamma^2) = 0$. Since $V(\varphi) = 0 = V(\gamma)$ for $V \perp R$, it follows that $e^{-\varphi}\gamma^2$ is constant on M^n . Hence,

$$(\mathcal{I} \circ g)_* = k f_* \hat{T}, \quad k \in \mathbb{R}. \quad (50)$$

On the other hand, from (37) we obtain that

$$\hat{T}(R - iS) = \lambda(R - iS),$$

and hence \hat{T} is an orthogonal tensor on M^n having only λ and $\bar{\lambda}$ as eigenvalues. Moreover, since $\mathcal{I} \circ g$ and f are homothetic by (50), it follows from Proposition 2 that \hat{T} is parallel. In particular, this implies that λ is constant on M^n and that $\hat{E}_\lambda = \ker(\hat{T} - \lambda I)$ is a parallel subbundle of $TM \otimes \mathbb{C}$. Hence, $\hat{J} \in \Gamma((TM \otimes \mathbb{C})^* \otimes (TM \otimes \mathbb{C}))$ defined by $\hat{J}Z = iZ$ for $Z \in \hat{E}_\lambda$ and $\hat{J}Z = -iZ$ for $Z \in \hat{E}_{\bar{\lambda}}$ is an almost complex structure on $TM \otimes \mathbb{C}$ that is parallel with respect to the complexified Levi-Civita connection of M^n . Since $\hat{J}(\bar{Z}) = \hat{J}Z$, it follows that \hat{J} comes from a parallel almost complex structure J on M^n that makes it a Kaehler manifold.

Using that the second fundamental form of f commutes with T , we obtain that it also commutes with \hat{T} , and hence with J . Thus f is a minimal real Kaehler cone. Finally, it follows from (50) that $f_\theta = \mathcal{I} \circ g$ is homothetic to a member of its associated family. Since $g = \mathcal{I} \circ f_\theta$, the proof is completed. ■

Proof of Theorem 22: The proof follows from Lemmas 30, 32, 33, 34, 35 and 36. ■

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IMPA – Estrada Dona Castorina, 110
 22460-320 – Rio de Janeiro – Brazil
 E-mail: marcos@impa.br

Universidade Federal de São Carlos
 13565-905 – São Carlos – Brazil
 E-mail: tojeiro@dm.ufscar.br